

# On $q$ -deformed infinite-dimensional $n$ -algebra

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## Abstract

The  $q$ -deformation of the infinite-dimensional  $n$ -algebra is investigated. Based on the structure of the  $q$ -deformed Virasoro-Witt algebra, we derive a nontrivial  $q$ -deformed Virasoro-Witt  $n$ -algebra which is nothing but a sh- $n$ -Lie algebra. Furthermore in terms of the pseudodifferential operators on the quantum plane, we construct the (co)sine  $n$ -algebra and the  $q$ -deformed  $SDiff(T^2)$   $n$ -algebra. We prove that they are the sh- $n$ -Lie algebras for the case of even  $n$ . An explicit physical realization of the (co)sine  $n$ -algebra is given.

KEYWORDS:  $q$ -deformation, Conformal and  $W$  Symmetry,  $n$ -algebra

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# 1 Introduction

Quantum algebras or more precisely quantized universal enveloping algebras first appeared in connection with the study of the inverse scattering problem. It is one parameter  $q$  deformation of Lie algebras which preserves the structure of a Hopf algebra and reduces to standard Lie algebra in the classical limit. The Virasoro algebra is an infinite dimensional Lie algebra and plays important roles in physics. Its  $q$ -deformation has been widely studied in the literature [1]-[10]. A  $q$ -deformation of the centerless Virasoro or Virasoro-Witt (V-W) algebra was first obtained by Curtright and Zachos [1]. Its central extension was later furnished by Aizawa and Sato [2]. Chaichian and Prešnajder [3] proposed a different version of the  $q$ -deformed Virasoro algebra by carrying out a Sugawara construction on a  $q$ -analogue of an infinite dimensional Heisenberg algebra. Shiraishi et al. [4] presented a  $q$ -Virasoro algebra  $Vir_{q,t}$ , where  $q$  and  $t$  are two complex parameters. They constructed a free boson realization of this  $q$ -Virasoro algebra and showed that singular vectors can be expressed by the Macdonald symmetric functions. It is similar as the case of the ordinary Virasoro algebra whose singular vectors are given by the Jack symmetric functions. It is well-known that there is a remarkable connection between the Virasoro algebra and the Korteweg-de Vries (KdV) equation [11, 12]. For the  $q$ -deformed Virasoro algebra, Chaichian et al. [13] showed that it generates the symplectic structure which can be used for a description of the discretization of the KdV equation. Furthermore the quantum KdV equations associated with the algebraic symmetry have been investigated in Refs.[14, 15]. The integrable one-dimensional quantum spin chains have attracted much interest from physical and mathematical points of view. One noted that the deformed Virasoro algebra plays an important role in the study of the XYZ model [16, 17].

The  $W_\infty$  algebra is the higher-spin extension of the Virasoro algebra. Its  $q$ -deformation has been constructed [6, 18]. The classical limit of the  $W_\infty$  algebra gives the so-called  $w_\infty$  algebra which is equivalent to the algebra of smooth area-preserving diffeomorphisms of the cylinder  $S^1 \times R^1$  [19]. It is worth to emphasize that the algebra of the area-preserving diffeomorphisms of the torus  $T^2$ , i.e.,  $SDiff(T^2)$  [20, 21] is also an important infinite-dimensional algebra. In terms of the Gauss derivatives on the quantum plane, Kinani et al. [22] presented the  $q$ -deformed  $SDiff(T^2)$  algebra. It should be noted that the sine algebra arises as the unique Lie algebra deformation of  $SDiff(T^2)$  in some suitable basis. There has been considerable interest in the (super) sine algebra [23]-[26].

The Nambu 3-algebra was introduced in [27, 28] as a natural generalization of a Lie algebra for higher-order algebraic operations. Recently Bagger and Lambert [29, 30], and Gustavsson [31] (BLG) found that 3-algebras play an important role in world-volume description of multiple M2-branes. Due to BLG theory, there has been considerable interest in the 3-algebra and its application. More recently there has been the progress in constructing the infinite-dimensional

3-algebras, such as V-W [32, 33], Kac-Moody [34] and  $w_\infty$  3-algebras [35, 36]. Moreover the relation between the infinite-dimensional 3-algebras and the integrable systems has also been studied. [37, 38].

Recently Curtright et al. [32], constructed a V-W algebra through the use of  $\mathfrak{su}(1,1)$  enveloping algebra techniques. It is worthwhile to mention that this ternary algebra depends on a parameter  $z$  and is only a Nambu-Lie algebra when  $z = \pm 2i$ . Ammar et al. [39] presented a  $q$ -deformation of this 3-algebra and noted it carrying the structure of ternary Hom-Nambu-Lie algebra. It is well known that the structure and property of  $q$ -deformed algebra are now very well understood. However for the  $q$ -deformed infinite-dimensional 3-algebra, much less is still known about its structure and property. As to the  $q$ -deformed infinite-dimensional  $n$ -algebra, to our best knowledge, it has not been reported so far in the existing literature. The goal of this paper is to construct the  $q$ -deformed infinite-dimensional  $n$ -algebras and explore their intriguing features.

This paper is organized as follows. In section 2, we introduce the definitions of  $n$ -Lie algebra and sh- $n$ -Lie algebra. In section 3, we construct the  $q$ -deformed V-W  $n$ -algebra. In section 4, in terms of the pseud-differential operators on the quantum plane, we construct the (co)sine  $n$ -algebra and the  $q$ -deformed  $SDiff(T^2)$   $n$ -algebra. An explicit physical realization of the (co)sine  $n$ -algebra is given in section 5. We end this paper with the concluding remarks in section 6.

## 2 $n$ -Lie algebra and sh- $n$ -Lie algebra

For later convenience, we shall recall the definitions of  $n$ -Lie algebra and sh- $n$ -Lie algebra in this section. For a more detailed description we refer the reader to Refs.[40]-[42].

The notion of  $n$ -Lie algebra or Filippov  $n$ -algebra was introduced by Filippov [40]. It is a natural generalization of Lie algebra.

**Definition 1** [40] *An  $n$ -Lie algebra structure is a linear space  $V$  endowed with a multilinear map called Nambu bracket  $[\cdot, \dots, \cdot]: V^{\otimes n} \rightarrow V$  satisfying the following properties:*

(1). *Skew-symmetry*

$$[A_{\sigma(1)}, \dots, A_{\sigma(n)}] = (-1)^{\epsilon(\sigma)} [A_1, \dots, A_n]. \quad (1)$$

(2). *Fundamental identity (FI) or Filippov condition*

$$[A_1, \dots, A_{n-1}, [B_1, \dots, B_n]] = \sum_{k=1}^n [B_1, \dots, B_{k-1}, [A_1, \dots, A_{n-1}, B_k], B_{k+1}, \dots, B_n]. \quad (2)$$

Recently 3-Lie algebra has attracted much interest due to its application in M-theory. For the case of 3-Lie algebra, the corresponding FI is

$$[A, B, [C, D, E]] = [[A, B, C], D, E] + [C, [A, B, D], E] + [C, D, [A, B, E]]. \quad (3)$$

We have already seen that an  $n$ -Lie algebra  $A$  is a vector space  $A$  endowed with an  $n$ -ary skew-symmetric multiplication satisfying the FI (2). We now turn to the notion of sh- $n$ -Lie algebra.

**Definition 2** [41] Let  $[\cdot, \dots, \cdot]$  be an  $n$ -ary skewsymmetric product on a vector space  $V$ . We say that  $(V, [\cdot, \dots, \cdot])$  is a sh- $n$ -Lie algebra if  $[\cdot, \dots, \cdot]$  satisfies the sh-Jacobi's identity

$$\sum_{\sigma \in Sh(n, n-1)} (-1)^{\epsilon(\sigma)} [[x_{\sigma(1)}, \dots, x_{\sigma(n)}], x_{\sigma(n+1)}, \dots, x_{\sigma(2n-1)}] = 0, \quad (4)$$

for any  $x_i \in A$ , where  $Sh(n, n-1)$  is the subset of  $\Sigma_{2n-1}$  defined by

$$Sh(n, n-1) = \{\sigma \in \Sigma_{2n-1}, \sigma(1) < \dots < \sigma(n), \sigma(n+1) < \dots < \sigma(2n-1)\}.$$

In terms of the Lévi-Civita symbol, i.e.,

$$\epsilon_{j_1 \dots j_p}^{i_1 \dots i_p} = \det \begin{pmatrix} \delta_{j_1}^{i_1} & \dots & \delta_{j_p}^{i_1} \\ \vdots & & \vdots \\ \delta_{j_1}^{i_p} & \dots & \delta_{j_p}^{i_p} \end{pmatrix}, \quad (5)$$

the sh-Jacobi's identity (4) can also be expressed as

$$\epsilon_{m_1 \dots m_{2n-1}}^{i_1 \dots i_{2n-1}} [[x_{i_1}, \dots, x_{i_n}], x_{i_{n+1}}, \dots, x_{i_{2n-1}}] = 0. \quad (6)$$

When  $n = 2$ , both the FI (2) and sh-Jacobi's identity (4) become the well-known Jacobi's identity. When  $n = 3$ , the FI is given by (3). For this case, the corresponding sh-Jacobi's identity (4) is

$$\begin{aligned} & [[A, B, C], D, E] - [[A, B, D], C, E] + [[A, B, E], C, D] + [[A, C, D], B, E] \\ & - [[A, C, E], B, D] + [[A, D, E], B, C] - [[B, C, D], A, E] + [[B, C, E], A, D] \\ & - [[B, D, E], A, C] + [[C, D, E], A, B] = 0. \end{aligned} \quad (7)$$

We have briefly introduced the  $n$ -Lie algebra and sh- $n$ -Lie algebra. It should be noted that any  $n$ -Lie algebra is a sh- $n$ -Lie algebra, but a sh- $n$ -Lie algebra is an  $n$ -Lie algebra if and only if any adjoint operator is a derivation.

### 3 $q$ -deformed V-W $n$ -algebra

#### 3.1 $q$ -deformed V-W 3-algebra

As a start before investigating the  $q$ -deformed 3-algebra, let us recall the case of  $q$ -deformed algebra. The deformation of the commutator is defined by

$$[A, B]_{(p,q)} = pAB - qBA. \quad (8)$$

It possesses the following properties [5, 10]:

$$\begin{aligned}
[A, B]_{(p,q)} &= -[B, A]_{(q,p)}, \\
[A + B, C]_{(p,q)} &= [A, C]_{(p,q)} + [B, C]_{(p,q)}, \\
[AB, C]_{(p,q)} &= A[B, C]_{(p,r)} + [A, C]_{(r,q)}B, \\
[A, BC]_{(p,q)} &= B[A, C]_{(r,q)} + [A, B]_{(p,r)}C,
\end{aligned} \tag{9}$$

and the  $q$ -Jacobi identity

$$\begin{aligned}
&[A, [B, C]_{(q_1, q_1^{-1})}]_{(q_3/q_2, q_2/q_3)} + [B, [C, A]_{(q_2, q_2^{-1})}]_{(q_1/q_3, q_3/q_1)} \\
&+ [C, [A, B]_{(q_3, q_3^{-1})}]_{(q_2/q_1, q_1/q_2)} = 0.
\end{aligned} \tag{10}$$

The Virasoro algebra is an infinite dimensional Lie algebra and plays important roles in physics. The V-W algebra is indeed the centerless Virasoro algebra. It is given by

$$[L_m, L_n] = (m - n)L_{m+n}. \tag{11}$$

To construct the deformed V-W algebra, let us take the  $q$ -deformed generators

$$L_m = -q^N (a^\dagger)^{m+1} a, \tag{12}$$

where the  $q$ -deformed oscillator is deformed by the following relations [43]-[45]:

$$\begin{aligned}
aa^\dagger - qa^\dagger a &= q^{-N}, \quad aa^\dagger = [N], \\
[N, a] &= -a, \quad [N, a^\dagger] = a^\dagger.
\end{aligned} \tag{13}$$

Substituting the  $q$ -generators (12) into the commutator (8) and using the  $q$ -deformed oscillator (13), it leads to the so-called  $q$ -deformed V-W algebra [1]

$$[L_m, L_n]_{(q^{m-n}, q^{n-m})} = q^{m-n} L_m L_n - q^{n-m} L_n L_m = [m - n] L_{m+n}, \tag{14}$$

where  $[k] = \frac{q^k - q^{-k}}{q - q^{-1}}$ . In the limit  $q \rightarrow 1$ , (14) reduces to the V-W algebra (11)

Let us define the star product by

$$\begin{aligned}
L_n * [L_m, L_k]_{(q^{m-k}, q^{k-m})} &= q^{2n-m-k} L_n [L_m, L_k]_{(q^{m-k}, q^{k-m})}, \\
[L_m, L_k]_{(q^{m-k}, q^{k-m})} * L_n &= q^{m+k-2n} [L_m, L_k]_{(q^{m-k}, q^{k-m})} L_n.
\end{aligned} \tag{15}$$

Then we have

$$\begin{aligned}
&L_n * [L_m, L_k]_{(q^{m-k}, q^{k-m})} - [L_m, L_k]_{(q^{m-k}, q^{k-m})} * L_n \\
&= [L_n, [L_m, L_k]_{(q^{m-k}, q^{k-m})}]_{(q^{2n-m-k}, q^{m+k-2n})}.
\end{aligned} \tag{16}$$

By means of (16), one can confirm the following  $q$ -Jacobi identity [5] satisfied by the  $q$ -deformed V-W algebra (14):

$$[L_n, [L_m, L_k]_{(q^{m-k}, q^{k-m})}]_{(q^{2n-m-k}, q^{m+k-2n})} + \text{cycl. perms.} = 0. \quad (17)$$

Let us now turn our attention to the case of 3-algebra. The operator Nambu 3-bracket is defined to be a sum of single operators multiplying commutators of the remaining two [27], i.e.,

$$[A, B, C] = A[B, C] + B[C, A] + C[A, B], \quad (18)$$

where  $[A, B] = AB - BA$ .

For the  $q$ -deformed V-W algebra (14), we have already seen that the  $q$ -Jacobi identity (17) is guaranteed to hold. It is worth to emphasize that the star product (15) plays a pivotal role in the  $q$ -Jacobi identity. In terms of the star product (15), let us define the  $q$ -3-bracket as follows:

$$\begin{aligned} \llbracket L_m, L_n, L_k \rrbracket &= L_m * [L_n, L_k]_{(q^{n-k}, q^{k-n})} + L_n * [L_k, L_m]_{(q^{k-m}, q^{m-k})} \\ &+ L_k * [L_m, L_n]_{(q^{m-n}, q^{n-m})}. \end{aligned} \quad (19)$$

By means of (14) and (15), we may derive the following  $q$ -deformed 3-algebra from (19):

$$\begin{aligned} \llbracket L_m, L_n, L_k \rrbracket &= \frac{1}{q - q^{-1}} ([2m - 2k] + [2k - 2n] + [2n - 2m]) L_{m+n+k} \\ &= (q - q^{-1}) ([m - n][m - k][n - k]) L_{m+n+k} \\ &= -\frac{1}{(q - q^{-1})^2} \det \begin{pmatrix} q^{-2m} & q^{-2n} & q^{-2k} \\ 1 & 1 & 1 \\ q^{2m} & q^{2n} & q^{2k} \end{pmatrix} L_{m+n+k}. \end{aligned} \quad (20)$$

Performing lengthy but straightforward calculations, we find that (20) satisfies the sh-Jacobi's identity (7), but the FI (3) does not hold. It is easy to verify that the skew-symmetry holds for this ternary algebra

$$\llbracket L_m, L_n, L_k \rrbracket = -\llbracket L_n, L_m, L_k \rrbracket = -\llbracket L_k, L_n, L_m \rrbracket.$$

Therefore the  $q$ -deformed V-W 3-algebra (20) is indeed a sh-3-Lie algebra. In the limit  $q \rightarrow 1$ , (20) reduces to the null 3-algebra derived in [33],

$$[L_m, L_n, L_k] = 0.$$

The FI (3) is trivially satisfied for this null 3-algebra.

### 3.2 $q$ -deformed V-W $n$ -algebra

Now encouraged by the possibility of constructing the nontrivial sh-3-Lie algebra (20), it would be interesting to study further and see whether one could construct the  $q$ -deformed V-W  $n$ -algebra with a genuine sh- $n$ -Lie algebra structure. In this section we give affirmative answer to this question.

The  $n$ -bracket with  $n \geq 3$  is defined by

$$[L_{i_1}, L_{i_2}, \dots, L_{i_n}] = \sum_{s=1}^n (-1)^{s+1} L_{i_s} [L_{i_1}, L_{i_2}, \dots, \widehat{L_{i_s}}, \dots, L_{i_n}]. \quad (21)$$

Here we denote a notational convention used frequently in the rest of this paper. Namely for any arbitrary symbol  $Z$ , the hat symbol  $\hat{Z}$  stands for the term that is omitted.

Let us define a  $q$ - $n$ -bracket as follows:

$$\llbracket L_{i_1}, L_{i_2}, \dots, L_{i_n} \rrbracket = \sum_{s=1}^n (-1)^{s+1} L_{i_s} * \llbracket L_{i_1}, L_{i_2}, \dots, \widehat{L_{i_s}}, \dots, L_{i_n} \rrbracket, \quad (22)$$

where the general star product is given by

$$L_{i_1} * \llbracket L_{i_2}, L_{i_3}, \dots, L_{i_n} \rrbracket = q^{xi_1+y(i_2+\dots+i_n)} L_{i_1} \llbracket L_{i_2}, L_{i_3}, \dots, L_{i_n} \rrbracket, \quad (23)$$

in which  $(x = 2, y = -1)$  for  $n = 3$ ,  $(x = n, y = 0)$  for even  $n \geq 4$  and  $(x = n - 1, y = -2)$  for odd  $n \geq 5$ . As done in the case of  $q$ -3-bracket (19), we introduce the general star product (23) into the  $q$ - $n$ -bracket here. It should be noted that the general star product (23) will play an important role in deriving the desired  $q$ -deformed V-W  $n$ -algebra.

**Theorem 3** *The  $q$ -generators (12) satisfy the following closed algebraic structure relation:*

$$\begin{aligned} \llbracket L_{i_1}, L_{i_2}, \dots, L_{i_n} \rrbracket &= \frac{\text{sign}(n)}{(q - q^{-1})^{n-1}} \\ &\det \begin{pmatrix} q^{-2\lfloor \frac{n-1}{2} \rfloor i_1} & q^{-2\lfloor \frac{n-1}{2} \rfloor i_2} & \dots & q^{-2\lfloor \frac{n-1}{2} \rfloor i_n} \\ q^{2(-\lfloor \frac{n-1}{2} \rfloor + 1)i_1} & q^{2(-\lfloor \frac{n-1}{2} \rfloor + 1)i_2} & \dots & q^{2(-\lfloor \frac{n-1}{2} \rfloor + 1)i_n} \\ \vdots & \vdots & \vdots & \vdots \\ q^{2(\lfloor \frac{n}{2} \rfloor - 1)i_1} & q^{2(\lfloor \frac{n}{2} \rfloor - 1)i_2} & \dots & q^{2(\lfloor \frac{n}{2} \rfloor - 1)i_n} \\ q^{2\lfloor \frac{n}{2} \rfloor i_1} & q^{2\lfloor \frac{n}{2} \rfloor i_2} & \dots & q^{2\lfloor \frac{n}{2} \rfloor i_n} \end{pmatrix} L_{\Sigma_{l=1}^n i_l}, \end{aligned} \quad (24)$$

where  $\lfloor n \rfloor = \text{Max}\{m \in \mathbb{Z} | m \leq n\}$  is the floor function,  $\text{sign}(n)$  is the signature function, i.e.,  $\text{sign}(n) = \begin{cases} 1, & \text{for } n \bmod 4 = 0, 1 \\ -1, & \text{for } n \bmod 4 = 2, 3 \end{cases}$ .

**Proof.** Let us confirm this by the mathematical induction for  $n$ . Equation (20) indicates that

(24) is satisfied for  $n = 3$ . We suppose (24) is satisfied for  $n$ . By means of (22), we have

$$\begin{aligned} & \llbracket L_{i_1}, L_{i_2}, \dots, L_{i_{n+1}} \rrbracket \\ &= \frac{\text{sign}(n)}{(q - q^{-1})^{n-1}} A \left[ q^{-\sum_{j=1}^{n+1} i_j - 1} q^{2N} (a^+)^{\sum_{j=1}^{n+1} i_j + 2} a^2 - \frac{1}{(q - q^{-1})} L_{\sum_{j=1}^{n+1} i_j} \right] + \frac{\text{sign}(n)}{(q - q^{-1})^n} \\ & \det \begin{pmatrix} q^{x_{i_1}} & \dots & q^{x_{i_s}} & \dots & q^{x_{i_{n+1}}} \\ q^{(y-2\lfloor \frac{n-1}{2} \rfloor) i_1} & \dots & q^{(y-2\lfloor \frac{n-1}{2} \rfloor) i_s} & \dots & q^{(y-2\lfloor \frac{n-1}{2} \rfloor) i_{n+1}} \\ q^{(y-2\lfloor \frac{n-1}{2} \rfloor + 2) i_1} & \dots & q^{(y-2\lfloor \frac{n-1}{2} \rfloor + 2) i_s} & \dots & q^{(y-2\lfloor \frac{n-1}{2} \rfloor + 2) i_{n+1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ q^{(y+2\lfloor \frac{n}{2} \rfloor) i_1} & \dots & q^{(y+2\lfloor \frac{n}{2} \rfloor) i_s} & \dots & q^{(y+2\lfloor \frac{n}{2} \rfloor) i_{n+1}} \end{pmatrix} L_{\sum_{l=1}^{n+1} i_l}, \quad (25) \end{aligned}$$

where  $A = \det \begin{pmatrix} q^{x_{i_1}} & \dots & q^{x_{i_s}} & \dots & q^{x_{i_{n+1}}} \\ q^{(y-2\lfloor \frac{n-1}{2} \rfloor + 2) i_1} & \dots & q^{(y-2\lfloor \frac{n-1}{2} \rfloor + 2) i_s} & \dots & q^{(y-2\lfloor \frac{n-1}{2} \rfloor + 2) i_{n+1}} \\ q^{(y-2\lfloor \frac{n-1}{2} \rfloor + 4) i_1} & \dots & q^{(y-2\lfloor \frac{n-1}{2} \rfloor + 4) i_s} & \dots & q^{(y-2\lfloor \frac{n-1}{2} \rfloor + 4) i_{n+1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ q^{(y+2\lfloor \frac{n}{2} \rfloor + 2) i_1} & \dots & q^{(y+2\lfloor \frac{n}{2} \rfloor + 2) i_s} & \dots & q^{(y+2\lfloor \frac{n}{2} \rfloor + 2) i_{n+1}} \end{pmatrix}.$

Substituting  $(x = n, y = -2)$  for even  $n$  and  $(x = n + 1, y = 0)$  for odd  $n$  into (25), respectively, we find that the determinate  $A$  is zero. After a straightforward calculation for the second determinate in (25), we obtain the explicit form of  $(n + 1)$ -bracket (25)

$$\begin{aligned} & \llbracket L_{i_1}, L_{i_2}, \dots, L_{i_{n+1}} \rrbracket = \frac{\text{sign}(n + 1)}{(q - q^{-1})^n} \\ & \det \begin{pmatrix} q^{-2\lfloor \frac{n}{2} \rfloor i_1} & q^{-2\lfloor \frac{n}{2} \rfloor i_2} & \dots & q^{-2\lfloor \frac{n}{2} \rfloor i_{n+1}} \\ q^{2(-\lfloor \frac{n}{2} \rfloor + 1) i_1} & q^{2(-\lfloor \frac{n}{2} \rfloor + 1) i_2} & \dots & q^{2(-\lfloor \frac{n}{2} \rfloor + 1) i_{n+1}} \\ \vdots & \vdots & \vdots & \vdots \\ q^{2(\lfloor \frac{n+1}{2} \rfloor - 1) i_1} & q^{2(\lfloor \frac{n+1}{2} \rfloor - 1) i_2} & \dots & q^{2(\lfloor \frac{n+1}{2} \rfloor - 1) i_{n+1}} \\ q^{2\lfloor \frac{n+1}{2} \rfloor i_1} & q^{2\lfloor \frac{n+1}{2} \rfloor i_2} & \dots & q^{2\lfloor \frac{n+1}{2} \rfloor i_{n+1}} \end{pmatrix} L_{\sum_{l=1}^{n+1} i_l}, \end{aligned}$$

which shows that (24) is satisfied for  $n + 1$ . Now the proof is completed.  $\blacksquare$

For the  $q$ -3-bracket (20), we already recognize that it satisfies the sh-Jacobi's identity (7), but the FI (2) does not hold. Let us consider the case of the  $q$ - $n$ -bracket (24). Taking  $A_i = L_{-i-1}, i = 1, 2, \dots, n - 2, A_{n-1} = L_{\frac{(n-1)n}{2}}$  and  $B_j = L_{j-1}, j = 1, 2, \dots, n$  in (2), straightforward calculation shows that the left-hand side of (2) equals zero, but its right-hand side does not. It indicates that the FI (2) does not hold for (24). Therefore the  $q$ - $n$ -bracket relation (24) is not an  $n$ -Lie algebra. In spite of this negative result it is instructive to pursue the analysis of the  $q$ - $n$ -bracket (24).

**Proposition 4** *When  $n \geq 3$ , the  $q$ - $n$ -bracket relation (24) is a sh- $n$ -Lie algebra.*



**Proof.** Let us first focus on (24) with odd  $n$ . In terms of the Lévi-Civita symbol (5), we can rewrite  $(2n+1)$ -bracket (24) as

$$[[L_{i_1}, \dots, L_{i_{2n+1}}]] = \frac{\text{sign}(2n+1)}{(q-q^{-1})^{2n}} \epsilon_{i_1 \dots i_{2n+1}}^{j_1 \dots j_{2n+1}} q^{-2nj_1+2(-n+1)j_2+\dots+2(n-1)j_{2n}+2nj_{2n+1}} L_{\sum_{l=1}^{2n+1} i_l}. \quad (26)$$

Then let us use the expression (26) to calculate  $[[[L_{i_1}, \dots, L_{i_{2n+1}}], L_{i_{2n+2}}, \dots, L_{i_{4n+1}}]]$ . It leads to

$$\begin{aligned} & [[L_{i_1}, \dots, L_{i_{2n+1}}], L_{i_{2n+2}}, \dots, L_{i_{4n+1}}]] \\ &= \sum_{k=2}^{2n+2} \frac{(-1)^k}{(q-q^{-1})^{4n}} \epsilon_{i_1 \dots i_{2n+1}}^{j_1 \dots j_{2n+1}} \epsilon_{i_{2n+2} \dots i_{4n+1}}^{j_{2n+2} \dots \widehat{j_{2n+k}} \dots j_{4n+2}} q^{2(-2n+k-2)j_1+2(-2n+k-1)j_2+\dots+2(k-2)j_{2n+1}} \\ & q^{-2nj_{2n+2}+\dots+2(-n+k-3)j_{2n+k-1}+2(-n+k-2)j_{2n+k}+2(-n+k-1)j_{2n+k+1}+\dots+2nj_{4n+2}} L_{\sum_{l=1}^{4n+1} i_l}. \end{aligned} \quad (27)$$

Substituting (27) into the left-hand side of (6), we obtain

$$\begin{aligned} & \epsilon_{m_1 \dots m_{4n+1}}^{i_1 \dots i_{4n+1}} [[L_{i_1}, \dots, L_{i_{2n+1}}], L_{i_{2n+2}}, \dots, L_{i_{4n+1}}]] \\ &= \frac{(2n+1)!(2n)!}{(q-q^{-1})^{4n}} \sum_{k=2}^{2n+2} (-1)^k \epsilon_{m_1 \dots m_{4n+1}}^{j_1 \dots \widehat{j_{2n+k}} \dots j_{4n+2}} q^\alpha L_{\sum_{l=1}^{4n+1} i_l}, \end{aligned} \quad (28)$$

where the power of  $q$  is given by

$$\begin{aligned} \alpha &= 2(-2n+k-2)j_1+2(-2n+k-1)j_2+\dots+2(k-2)j_{2n+1} \\ & -2nj_{2n+2}+\dots+2(-n+\widehat{k-2})j_{2n+k}+\dots+2nj_{4n+2}, \end{aligned} \quad (29)$$

and the following formula is useful in simplifying expression:

$$\epsilon_{m_1 \dots m_n}^{i_1 \dots i_n} \epsilon_{i_1 \dots i_k}^{j_1 \dots j_k} = k! \epsilon_{m_1 \dots m_n}^{j_1 \dots j_k i_{k+1} \dots i_n}. \quad (30)$$

From the expression of  $\alpha$  (29), we observe that the coefficients of two different  $j_\mu$  should be equal. Since  $\epsilon_{1 \dots 4n+1}^{j_1 \dots \widehat{j_{2n+k}} \dots j_{4n+2}}$  is completely antisymmetric, it is easy to see that (28) equals zero. It indicates that the sh-Jacobi's identity is satisfied by (24) with odd  $n$ .

For the case of (24) with even  $n$ , by the similar way, we can confirm the corresponding sh-Jacobi's identity. Taking the above results, we may conclude that the sh-Jacobi's identity (4) does hold for (24). Since the structure constants are determined by the determinant,  $n$ -bracket (24) is anticommutative. Based on the above analysis, it is clear that the  $q$ -deformed V-W  $n$ -algebra is indeed a sh- $n$ -Lie algebra.  $\blacksquare$

We have constructed the  $q$ -deformed V-W  $n$ -algebra (24). It should be noted that the structure constant of this  $q$ -deformed infinite-dimensional  $n$ -algebra is determined by the Vandermonde determinant. In the limit  $q \rightarrow 1$ , it is easy to see that (24) reduces to the null  $n$ -algebra. It is also interesting to note that the structure constant of the  $q$ -deformed V-W  $(n-1)$ -algebra

can be induced from that of (24). More precisely, it is equal to  $(-1)^{n-1} (q - q^{-1})$  times the structure constant of (24), where the original Vandermonde determinant in (24) is replaced by the  $(h(n), n)$ -minor of its Vandermonde matrix,  $h(n)$  takes 1 and  $n$  for odd and even  $n$ , respectively.

Let us list first few  $q$ -deformed V-W  $n$ -algebras as follows:

$$\begin{aligned}
& \bullet \quad \llbracket L_{i_1}, L_{i_2}, L_{i_3}, L_{i_4} \rrbracket \\
&= (q - q^{-1})^{-3} \det \begin{pmatrix} q^{-2i_1} & q^{-2i_2} & q^{-2i_3} & q^{-2i_4} \\ 1 & 1 & 1 & 1 \\ q^{2i_1} & q^{2i_2} & q^{2i_3} & q^{2i_4} \\ q^{4i_1} & q^{4i_2} & q^{4i_3} & q^{4i_4} \end{pmatrix} L_{\sum_{k=1}^4 i_k} \\
&= (q - q^{-1})^3 q^{\sum_{k=1}^4 i_k} \prod_{1 \leq m < n \leq 4} [i_m - i_n] L_{\sum_{k=1}^4 i_k}. \tag{31}
\end{aligned}$$

$$\begin{aligned}
& \bullet \quad \llbracket L_{i_1}, L_{i_2}, L_{i_3}, L_{i_4}, L_{i_5} \rrbracket \\
&= (q - q^{-1})^{-4} \det \begin{pmatrix} q^{-4i_1} & q^{-4i_2} & q^{-4i_3} & q^{-4i_4} & q^{-4i_5} \\ q^{-2i_1} & q^{-2i_2} & q^{-2i_3} & q^{-2i_4} & q^{-2i_5} \\ 1 & 1 & 1 & 1 & 1 \\ q^{2i_1} & q^{2i_2} & q^{2i_3} & q^{2i_4} & q^{2i_5} \\ q^{4i_1} & q^{4i_2} & q^{4i_3} & q^{4i_4} & q^{4i_5} \end{pmatrix} L_{\sum_{k=1}^5 i_k} \\
&= (q - q^{-1})^6 q^{\sum_{k=1}^5 i_k} \prod_{1 \leq m < n \leq 5} [i_m - i_n] L_{\sum_{k=1}^5 i_k}. \tag{32}
\end{aligned}$$

$$\begin{aligned}
& \bullet \quad \llbracket L_{i_1}, L_{i_2}, L_{i_3}, L_{i_4}, L_{i_5}, L_{i_6} \rrbracket \\
&= -(q - q^{-1})^{-5} \det \begin{pmatrix} q^{-4i_1} & q^{-4i_2} & q^{-4i_3} & q^{-4i_4} & q^{-4i_5} & q^{-4i_6} \\ q^{-2i_1} & q^{-2i_2} & q^{-2i_3} & q^{-2i_4} & q^{-2i_5} & q^{-2i_6} \\ 1 & 1 & 1 & 1 & 1 & 1 \\ q^{2i_1} & q^{2i_2} & q^{2i_3} & q^{2i_4} & q^{2i_5} & q^{2i_6} \\ q^{4i_1} & q^{4i_2} & q^{4i_3} & q^{4i_4} & q^{4i_5} & q^{4i_6} \\ q^{6i_1} & q^{6i_2} & q^{6i_3} & q^{6i_4} & q^{6i_5} & q^{6i_6} \end{pmatrix} L_{\sum_{k=1}^6 i_k} \\
&= -(q - q^{-1})^{10} q^{\sum_{k=1}^6 i_k} \prod_{1 \leq m < n \leq 6} [i_m - i_n] L_{\sum_{k=1}^6 i_k}. \tag{33}
\end{aligned}$$

## 4 $q$ -deformed $SDiff(T^2)$ $n$ -algebra

### 4.1 Sine 3-algebra and $q$ -deformed $SDiff(T^2)$ 3-algebra

The quantum differential calculus on the quantum plane  $\mathbf{C}_q[x, y]$  have been well investigated [46]. For the the quantum plane  $\mathbf{C}_q[x, y]$ , each of its elements is a finite linear combination of

the monomes  $y^n x^m$ , satisfying

$$x^m y^n = q^{nm} y^n x^m, \quad m, n \in \mathbf{N}. \quad (34)$$

The Gauss derivatives on  $\mathbf{C}_q[x, y]$  can be extended to be formal pseud-differential operators  $D_x, D_y$  which can be defined on the set  $\mathbf{C}_q[[x, y]]$  and satisfy

$$D_x^n D_y^m = q^{nm} D_y^m D_x^n, \quad m, n \in \mathbf{Z}, \quad (35)$$

where  $\mathbf{C}_q[[x, y]]$  is a set of all Laurent series in  $x, y$  such that (34) is valid for  $n, m \in \mathbf{Z}$ .

In terms of the pseud-differential operators  $D_x$  and  $D_y$ , Kinani et al. [22] introduced the following generators:

$$T_n = q^{n_1 \cdot n_2 / 2} \cdot D_y^{n_1} D_x^{n_2}, \quad (36)$$

where  $n = (n_1, n_2) \in \mathbf{Z}^2$ . In the rest of this paper, we denote the subscript  $l$  on  $T_l$  being a two-dimensional vector with integer components.

By means of (35), it is easy to verify that the generators  $T_n$  (36) satisfy

$$T_n T_m = q^{\frac{1}{2} m \wedge n} T_{n+m}, \quad (37)$$

where  $m \wedge n = m_1 n_2 - m_2 n_1$ .

Thus we have the algebra

$$[T_m, T_n] = T_m T_n - T_n T_m = (q^{\frac{1}{2} n \wedge m} - q^{\frac{1}{2} m \wedge n}) T_{m+n}. \quad (38)$$

When  $q = \exp(-2\pi i \alpha)$ , (38) becomes the sine algebra [23]

$$[T_m, T_n] = 2i \sin(\pi \alpha m \wedge n) T_{m+n}, \quad (39)$$

where  $\alpha$  is an arbitrary constant.

Taking the rescaled generators  $\bar{T}_m = -\frac{i}{2\pi\alpha} T_m$ , we note that in the limit  $\alpha \rightarrow 0$ , (39) leads to the  $SDiff(T^2)$  algebra [20, 21]

$$[\bar{T}_m, \bar{T}_n] = (m \wedge n) \bar{T}_{m+n}. \quad (40)$$

The  $q$ -deformation of  $SDiff(T^2)$  algebra (40) is given by [22]

$$[\bar{T}_m, \bar{T}_n]_{(q^{\frac{3}{2} m \wedge n}, q^{\frac{3}{2} n \wedge m})} = q^{\frac{3}{2} m \wedge n} \bar{T}_m \bar{T}_n - q^{\frac{3}{2} n \wedge m} \bar{T}_n \bar{T}_m = [m \wedge n] \bar{T}_{m+n}, \quad (41)$$

where  $\bar{T}_m = \frac{1}{q-q^{-1}} T_m$ .

Let us turn to the case of 3-algebra. Substituting the generators (36) into the operator Nambu 3-bracket (18) and using (37) and (38), by direct calculation, we may derive the following 3-algebra:

$$\begin{aligned} [T_m, T_n, T_k] &= (-q^{\frac{1}{2}(m \wedge n - n \wedge k + k \wedge m)} + q^{-\frac{1}{2}(m \wedge n - n \wedge k + k \wedge m)} \\ &\quad - q^{\frac{1}{2}(m \wedge n + n \wedge k - k \wedge m)} + q^{-\frac{1}{2}(m \wedge n + n \wedge k - k \wedge m)} \\ &\quad - q^{\frac{1}{2}(-m \wedge n + n \wedge k + k \wedge m)} + q^{-\frac{1}{2}(-m \wedge n + n \wedge k + k \wedge m)}) T_{m+n+k}. \end{aligned} \quad (42)$$

Performing straightforward calculations, we find that the 3-algebra with the  $q$  parameter (42) does not satisfy the FI (3) and the sh-Jacobi's identity (7).

An interesting case is for the special value of  $q$ . Taking  $q = e^{-\pi i}$ , we may rewrite (42) as

$$\begin{aligned} [T_m, T_n, T_k] &= 2i(\sin(\frac{\pi}{2}(m \wedge n - n \wedge k + k \wedge m)) + \sin(\frac{\pi}{2}(m \wedge n + n \wedge k - k \wedge m)) \\ &+ \sin(\frac{\pi}{2}(-m \wedge n + n \wedge k + k \wedge m)))T_{m+n+k}. \end{aligned} \quad (43)$$

Not as the case of (42), an intriguing property of (43) is that it does satisfy the FI (3). Since the skew symmetry also holds, the sine 3-algebra (43) is indeed a Fillipov 3-algebra.

Let us take the rescaled generators  $\bar{T}_n = \frac{1}{(q-q^{-1})^{1/2}}T_n$  and define the  $q$ -3-bracket

$$\begin{aligned} \llbracket \bar{T}_m, \bar{T}_n, \bar{T}_k \rrbracket &= \bar{T}_m * [\bar{T}_n, \bar{T}_k]_{(q^{\frac{3}{2}n \wedge k}, q^{\frac{3}{2}k \wedge n})} + \bar{T}_n * [\bar{T}_k, \bar{T}_m]_{(q^{\frac{3}{2}k \wedge m}, q^{\frac{3}{2}m \wedge k})} \\ &+ \bar{T}_k * [\bar{T}_m, \bar{T}_n]_{(q^{\frac{3}{2}m \wedge n}, q^{\frac{3}{2}n \wedge m})}, \end{aligned} \quad (44)$$

where the star product is given by

$$\bar{T}_m * [\bar{T}_n, \bar{T}_k]_{(q^{\frac{3}{2}n \wedge k}, q^{\frac{3}{2}k \wedge n})} = q^{\frac{3}{2}m \wedge (n+k)} \bar{T}_m [\bar{T}_n, \bar{T}_k]_{(q^{\frac{3}{2}n \wedge k}, q^{\frac{3}{2}k \wedge n})}. \quad (45)$$

Then we have the  $q$ -deformed  $SDiff(T^2)$  3-algebra

$$\begin{aligned} \llbracket \bar{T}_m, \bar{T}_n, \bar{T}_k \rrbracket &= ([m \wedge n - n \wedge k + k \wedge m] + [m \wedge n + n \wedge k - k \wedge m] \\ &+ [-m \wedge n + n \wedge k + k \wedge m])\bar{T}_{m+n+k} \\ &= ([\det \begin{pmatrix} m_1 & n_1 & k_1 \\ m_2 & n_2 & k_2 \\ -1 & 1 & 1 \end{pmatrix}] + [\det \begin{pmatrix} m_1 & n_1 & k_1 \\ m_2 & n_2 & k_2 \\ 1 & -1 & 1 \end{pmatrix}]) \\ &+ [\det \begin{pmatrix} m_1 & n_1 & k_1 \\ m_2 & n_2 & k_2 \\ 1 & 1 & -1 \end{pmatrix}])\bar{T}_{m+n+k}. \end{aligned} \quad (46)$$

As the case of (42), the infinite-dimensional  $q$ -deformed 3-algebra (46) does not satisfy the FI (3) and the sh-Jacobi's identity (7).

In the limit  $q \rightarrow 1$ , (46) reduces to the  $SDiff(T^2)$  3-algebra [37]

$$[\bar{T}_m, \bar{T}_n, \bar{T}_k] = (m \wedge n + n \wedge k + k \wedge m) \bar{T}_{m+n+k}. \quad (47)$$

The FI (3) is satisfied for this infinite-dimensional 3-algebra. Taking  $\bar{T}_k = \bar{T}_0$  in (47), (47) can be regarded as the parametrized bracket relation  $[\bar{T}_m, \bar{T}_n]_{\bar{T}_0}$ . This parametrized bracket relation gives rise to the  $SDiff(T^2)$  algebra (40).

## 4.2 (co)Sine $n$ -algebra

For the generators  $T_n$  (36), we note that they are the associative operators with the product (37). According to the definition of the  $n$ -bracket (21), we get the following result.

**Theorem 5** *The generators (36) satisfy the following closed algebraic structure relation:*

$$[T_{i_1}, \dots, T_{i_n}] = \epsilon_{i_1 \dots i_n}^{j_1 \dots j_n} q^{\frac{1}{2} \sum_{k>s} j_k \wedge j_s} T_{\sum_{l=1}^n i_l}. \quad (48)$$

**Proof.** The  $n$ -bracket (48) will follow from (37) if we can show that

$$[T_{i_1}, \dots, T_{i_n}] = \epsilon_{i_1 \dots i_n}^{j_1 \dots j_n} T_{j_1} T_{j_2} \dots T_{j_n}. \quad (49)$$

First, let us prove (49) by the mathematical induction for  $n$ . By (38), it is obvious that (49) holds for  $n = 2$ . We suppose (49) is satisfied for  $n$ -bracket. Note that the generators  $T_i$  are the associative operators under the product (37), we obtain

$$\begin{aligned} [T_{i_1}, \dots, T_{i_{n+1}}] &= \sum_{l=1}^{n+1} (-1)^{l-1} T_{i_l} [T_{i_1}, \dots, \hat{T}_{i_l}, \dots, T_{i_{n+1}}] \\ &= \sum_{l=1}^{n+1} (-1)^{l-1} \epsilon_{i_1 \dots \hat{i}_l \dots i_{n+1}}^{j_2 \dots j_{n+1}} T_{i_l} (T_{j_2} \dots T_{j_{n+1}}) \\ &= \sum_{l=1}^{n+1} (-1)^{l-1} \epsilon_{i_1 \dots \hat{i}_l \dots i_{n+1}}^{j_2 \dots j_{n+1}} \left( \delta_{i_l}^{j_1} T_{j_1} \right) (T_{j_2} \dots T_{j_{n+1}}) \\ &= \left( \sum_{l=1}^{n+1} (-1)^{l-1} \epsilon_{i_1 \dots \hat{i}_l \dots i_{n+1}}^{j_2 \dots j_{n+1}} \delta_{i_l}^{j_1} \right) T_{j_1} T_{j_2} \dots T_{j_{n+1}} \\ &= \epsilon_{i_1 \dots i_{n+1}}^{j_1 \dots j_{n+1}} T_{j_1} T_{j_2} \dots T_{j_{n+1}}, \end{aligned} \quad (50)$$

which shows that (49) is also satisfied for  $n + 1$ -bracket.

Substituting (37) into (49), we obtain (48). The proof is completed.  $\blacksquare$

When  $n = 3$  in (48), we have known that the corresponding 3-algebra (42) does not satisfy the FI (3) and the sh-Jacobi's identity (7). Let us now analyze the property of  $n$ -algebra (48) for  $n \geq 4$ .

**Proposition 6** *When  $n$  is even, the  $n$ -algebra (48) is a sh- $n$ -Lie algebra.*

**Proof.** Due to the skew-symmetry of  $\epsilon_{i_1 \dots i_n}^{j_1 \dots j_n}$  in (48), it is obvious that the skew-symmetry holds for the  $n$ -bracket (48). We are left to show that the sh-Jacobi's identity

$$\epsilon_{k_1 \dots k_{2n-1}}^{i_1 \dots i_{2n-1}} [[T_{i_1}, \dots, T_{i_n}], T_{i_{n+1}}, \dots, T_{i_{2n-1}}] = 0 \quad (51)$$

is satisfied.

Substituting (48) and (49) into the left-hand side of the sh-Jacobi's identity (51), we obtain

$$\begin{aligned}
& \epsilon_{k_1 \dots k_{2n-1}}^{i_1 \dots i_{2n-1}} \epsilon_{i_1 \dots i_n}^{j_1 \dots j_n} q^{\frac{1}{2} \sum_{1 \leq s < k \leq n} j_k \wedge j_s} [T_{\sum_{m=1}^n i_m}, T_{i_{n+1}}, \dots, T_{i_{2n-1}}] \\
&= \sum_{l=0}^{n-1} (-1)^l \epsilon_{k_1 \dots k_{2n-1}}^{i_1 \dots i_{2n-1}} \epsilon_{i_1 \dots i_n}^{j_1 \dots j_n} \epsilon_{\sum_{k=1}^n i_k, i_{n+1}, \dots, i_{2n-1}}^{j_{n+1} \dots j_{n+l}, \sum_{m=1}^n i_m, j_{n+l+2}, \dots, j_{2n}} \\
& \quad q^{\frac{1}{2} \sum_{1 \leq s < k \leq n} j_k \wedge j_s} T_{j_{n+1}} \dots T_{j_{n+l}} (T_{\sum_{k=1}^n i_k}) T_{j_{n+l+2}} \dots T_{j_{2n-1}} \\
&= \sum_{l=0}^{n-1} (-1)^l \epsilon_{k_1 \dots k_{2n-1}}^{i_1 \dots i_{2n-1}} \epsilon_{i_1 \dots i_n}^{j_1 \dots j_n} \epsilon_{i_{n+1} \dots i_{2n-1}}^{j_{n+1} \dots j_{2n-1}} T_{j_{n+1}} \dots T_{j_{n+l}} (T_{j_1} \dots T_{j_n}) T_{j_{n+l+1}} \dots T_{j_{2n-1}} \\
&= \sum_{l=0}^{n-1} (-1)^{l+ln} n! (n-1)! \epsilon_{k_1 \dots k_{2n-1}}^{j_1 \dots j_{2n-1}} T_{j_1} \dots T_{j_{2n-1}} \\
&= n! (n-1)! \epsilon_{k_1 \dots k_{2n-1}}^{j_1 \dots j_{2n-1}} q^{\frac{1}{2} \sum_{1 \leq s < k \leq 2n-1} j_k \wedge j_s} \sum_{l=0}^{n-1} (-1)^{l(n+1)} T_{\sum_{m=1}^{2n-1} k_m}. \tag{52}
\end{aligned}$$

Since  $\sum_{l=0}^{n-1} (-1)^{l(n+1)} = 0$  with even  $n$ , the right-hand side of (52) equals zero. Therefore the sh-Jacobi's identity (51) holds for even  $n$ . The proof is completed.  $\blacksquare$

We finally remark that when  $n$  is odd, the  $n$ -algebra (48) is not a sh- $n$ -Lie algebra. For (52) with odd  $n$ , the coefficient of  $T_{\sum_{m=1}^{2n-1} k_m}$  is

$$(n!)^2 \epsilon_{k_1 \dots k_{2n-1}}^{j_1 \dots j_{2n-1}} q^{\frac{1}{2} \sum_{1 \leq s < k \leq 2n-1} j_k \wedge j_s}. \tag{53}$$

Let us choose  $k_l = (l, 1)$ , we note that the coefficient of the monomial with the maximal power is

$$(n!)^2 \epsilon_{k_1 \dots k_{2n-1}}^{k_1 \dots k_{2n-1}} = (n!)^2, \tag{54}$$

It is obvious that the sh-Jacobi's identity does not hold for this case.

By the similar way, we can confirm that the  $n$ -algebra (48) does not also satisfy the FI (2).

We have derived the  $n$ -bracket (48) with general  $q \in \mathbf{C}$ . Let us now focus on the case of the special value of  $q$ . Taking  $q = \exp(-2\pi i \alpha)$ ,  $\alpha \in R$ , we can express the  $n$ -bracket (48) as

$$\begin{aligned}
& [T_{i_1}, \dots, T_{i_n}] \\
&= \frac{1}{2} \left( \epsilon_{i_1 \dots i_n}^{j_1 \dots j_n} \exp(\pi i \alpha \sum_{k < s} j_k \wedge j_s) + \epsilon_{i_1 \dots i_n}^{j_n \dots j_1} \exp(\pi i \alpha \sum_{k > s} j_k \wedge j_s) \right) T_{\sum_{l=1}^n i_l} \\
&= \frac{1}{2} \epsilon_{i_1 \dots i_n}^{j_1 \dots j_n} \cos(\pi \alpha \sum_{k < s} j_k \wedge j_s) \left( 1 + (-1)^{\frac{n(n-1)}{2}} \right) T_{\sum_{l=1}^n i_l} \\
& \quad + \frac{i}{2} \epsilon_{i_1 \dots i_n}^{j_1 \dots j_n} \sin(\pi \alpha \sum_{k < s} j_k \wedge j_s) \left( 1 - (-1)^{\frac{n(n-1)}{2}} \right) T_{\sum_{l=1}^n i_l}. \tag{55}
\end{aligned}$$

When  $n$  is even, (55) is a sh- $n$ -Lie algebra. However not as the case of (48) with odd  $n$ , we find that when  $n = 3$ , for the special value  $\alpha = \frac{1}{2}$ , (55) gives a Fillipov 3-algebra (43).

Let us consider the case of  $n = 5$ . In this case, (55) gives

$$[T_{i_1}, \dots, T_{i_5}] = \epsilon_{i_1 \dots i_5}^{j_1 \dots j_5} \cos(\pi \alpha \sum_{k < l} j_k \wedge j_l) T_{i_1 + \dots + i_5}. \tag{56}$$

Taking  $\alpha = \frac{1}{3}$  in (56), it is interesting to note that the sh-Jacobi's identity (4) holds, but the FI (2) fails in this example. Thus for this special  $\alpha$ , the cosine 5-algebra (56) gives a sh-5-Lie algebra.

### 4.3 $q$ -deformed $SDiff(T^2)$ $n$ -algebra

To construct the  $q$ -deformed  $SDiff(T^2)$   $n$ -algebra, let us define a  $q$ - $n$ -bracket as follows:

$$[\bar{T}_{i_1}, \bar{T}_{i_2}, \dots, \bar{T}_{i_n}] = \sum_{s=1}^n (-1)^{s+1} \bar{T}_{i_s} * [\bar{T}_{i_1}, \bar{T}_{i_2}, \dots, \hat{\bar{T}}_{i_s}, \dots, \bar{T}_{i_n}], \quad (57)$$

where the generators are  $\bar{T}_n = \frac{1}{(q-q^{-1})^{1/(n-1)}} T_n$  and the general star product is given by

$$\bar{T}_{i_1} * [\bar{T}_{i_2}, \bar{T}_{i_3}, \dots, \bar{T}_{i_n}] = q^{\frac{3}{2}i_1 \wedge (i_2 + \dots + i_n)} \bar{T}_{i_1} [\bar{T}_{i_2}, \bar{T}_{i_3}, \dots, \bar{T}_{i_n}], \quad (58)$$

According to the definition of the  $n$ -bracket (57), in similarity with the case of (48), we may derive the following  $q$ -deformed  $SDiff(T^2)$   $n$ -algebra:

$$[\bar{T}_{i_1}, \dots, \bar{T}_{i_n}] = \frac{\epsilon_{i_1 \dots i_n}^{j_1 \dots j_n} q^{\sum_{k < s} j_k \wedge j_s}}{q - q^{-1}} \bar{T}_{\sum_{l=1}^n i_l}. \quad (59)$$

When  $n = 3$ , (59) gives the  $q$ -deformed  $SDiff(T^2)$  3-algebra (46). In the limit  $q \rightarrow 1$ , it is not hard to verify that (59) reduces to the null  $n$ -algebra for  $n \geq 4$ ,

$$[\bar{T}_{i_1}, \dots, \bar{T}_{i_n}] = 0. \quad (60)$$

**Proposition 7** *When  $n$  is even, the  $n$ -algebra (59) is a sh- $n$ -Lie algebra.*

**Proof.** Due to the skew-symmetry of  $\epsilon_{i_1 \dots i_n}^{j_1 \dots j_n}$  in (59), it is obvious that the skew-symmetry holds for the  $n$ -bracket (59). We are left to show that the sh-Jacobi's identity is satisfied.

Substituting (59) into the left-hand side of (51) and using the formula (30), we get

$$\begin{aligned} & \frac{\epsilon_{k_1 \dots k_{2n-1}}^{i_1 \dots i_{2n-1}} \epsilon_{i_1 \dots i_n}^{j_1 \dots j_n} q^{\sum_{1 \leq k < l \leq n} j_k \wedge j_l}}{q - q^{-1}} [\bar{T}_{\sum_{k=1}^n j_k}, \bar{T}_{i_{n+1}}, \dots, \bar{T}_{i_{2n-1}}] \\ &= \frac{1}{(q - q^{-1})^2} \sum_{s=0}^{n-1} (-1)^s \epsilon_{k_1 \dots k_{2n-1}}^{i_1 \dots i_{2n-1}} \epsilon_{i_1 \dots i_n}^{j_1 \dots j_n} \epsilon_{i_{n+1} \dots i_{2n-1}}^{j_{n+1} \dots j_{2n-1}} q^{\sum_{1 \leq k < l \leq n} (j_k \wedge j_l)} q^{\sum_{n+1 \leq k < l \leq 2n-1} (j_k \wedge j_l)} \\ & \quad \cdot q^{\sum_{n+1 \leq l \leq n+s} j_l \wedge (\sum_{k=1}^n j_k)} q^{\sum_{n+s+1 \leq l \leq 2n-1} (\sum_{k=1}^n j_k) \wedge j_l} \bar{T}_{\sum_{k=1}^{2n-1} i_k} \\ &= \frac{1}{(q - q^{-1})^2} \sum_{s=0}^{n-1} (-1)^s n! (n-1)! \epsilon_{k_1 \dots k_{2n-1}}^{j_1 \dots j_{2n-1}} q^{\sum_{1 \leq k < l \leq n} (j_k \wedge j_l)} q^{\sum_{n+1 \leq k < l \leq 2n-1} (j_k \wedge j_l)} \\ & \quad \cdot q^{\sum_{n+1 \leq l \leq n+s} j_l \wedge (\sum_{k=1}^n j_k)} q^{\sum_{n+s+1 \leq l \leq 2n-1} (\sum_{k=1}^n j_k) \wedge j_l} \bar{T}_{\sum_{k=1}^{2n-1} i_k}. \end{aligned} \quad (61)$$

Let us change the indices  $(j_1, \dots, j_{2n-1})$  to be  $(j_{s+1}, \dots, j_{s+n}, j_1, \dots, j_s, j_{n+s+1}, \dots, j_{2n-1})$ , thus the right-hand side of (61) can be rewritten as

$$\begin{aligned} & \frac{1}{(q - q^{-1})^2} \sum_{s=0}^{n-1} (-1)^s n! (n-1)! \epsilon_{k_1 \dots k_{2n-1}}^{j_{s+1} \dots j_{s+n} j_1 \dots j_s j_{n+s+1} \dots j_{2n-1}} q^{\sum_{1 \leq k < l \leq 2n-1} j_k \wedge j_l} T_{\sum_{k=1}^{2n-1} i_k} \\ &= \frac{1}{(q - q^{-1})^2} n! (n-1)! \epsilon_{k_1 \dots k_{2n-1}}^{j_1 \dots j_{2n-1}} q^{\sum_{1 \leq k < l \leq 2n-1} j_k \wedge j_l} \sum_{s=0}^{n-1} (-1)^{s(n+1)} T_{\sum_{k=1}^{2n-1} i_k}. \end{aligned} \quad (62)$$

When  $n$  is even, we have  $\sum_{s=0}^{n-1} (-1)^{s(n+1)} = 0$ . It indicates that (61) equals zero. Therefore the sh-Jacobi's identity holds. The proof is completed.  $\blacksquare$

As the case of (48), it is easy to check that when  $n$  is odd, the  $n$ -algebra (59) does not satisfy the sh-Jacobi's identity. That is to say that (59) with odd  $n$  is not a sh- $n$ -Lie algebra.

## 5 A physical realization of the (co)sine $n$ -algebra

Let consider a spinless non-relativistic electron moving on a Bravais lattice in the  $xy$ -plane under the influence of a constant uniform magnetic field  $\mathbf{B} = B\mathbf{e}_z$ . The Hamiltonian is [25]

$$H = \frac{1}{2\mu} (\pi_x^2 + \pi_y^2) + V(x, y), \quad (63)$$

where the substrate potential  $V(x, y)$  is periodic in  $x$  and  $y$ , i.e.,  $V(x + a_1, y) = V(x, y + a_2) = V(x, y)$ , with  $a_1$  and  $a_2$  being the unit lattice spacing, the kinetic momentum operators are define by

$$\pi_x = p_x - \frac{e}{c} A_x, \quad \pi_y = p_y - \frac{e}{c} A_y, \quad (64)$$

in which  $p_x = -i\hbar \frac{\partial}{\partial x}$  and  $p_y = -i\hbar \frac{\partial}{\partial y}$  are the canonical momentum operators,  $\mathbf{A} = (A_x, A_y)$  is the vector potential and can be given by

$$A_x = -\frac{B}{2}y + \frac{\partial \Lambda}{\partial x}, \quad A_y = -\frac{B}{2}x + \frac{\partial \Lambda}{\partial y}. \quad (65)$$

Here  $\Lambda$  is an arbitrary scalar function determining the gauge. For simplicity, we choose  $\Lambda = \frac{1}{2}Bxy$ .

We take an arbitrary Bravais lattice vector as follows:

$$\mathbf{R}_m = m_1 \mathbf{a}_1 + m_2 \mathbf{a}_2, \quad (66)$$

where  $m = (m_1, m_2) \in \mathbf{Z}^2$ ,  $\mathbf{a}_1$  and  $\mathbf{a}_2$  are two given vectors in the directions  $\mathbf{e}_x$  and  $\mathbf{e}_y$ , respectively.

Let  $\boldsymbol{\beta} = (\beta_1, \beta_2)$  be a vector which is classically connected with the cyclotron center given by

$$\beta_1 = \pi_x - \mu\omega y, \quad \beta_2 = \pi_y + \mu\omega x, \quad (67)$$



where  $\omega = eB/\mu c$  is the Larmor frequency.

Let us take the magnetic translation operators [25]

$$T_m = \exp(\sqrt{2\pi} \mathbf{i} \mathbf{R}_m \cdot \boldsymbol{\beta} / \hbar). \quad (68)$$

Note that they satisfy

$$T_m T_n = T_{m+n} \exp(\pi \mathbf{i} \alpha m \wedge n), \quad (69)$$

where  $\alpha = \phi_1/\phi_0$  is the number of fluxons passing through the unit cell, in which  $\phi_1 = (a_1 \wedge a_2) \cdot B$ ,  $\phi_0 = hc/e$  are the magnetic flux through the unit cell  $a_1 \wedge a_2$ , and the magnetic flux quantum, respectively.

For the magnetic translation operators (68), it was found that they generate the infinite-dimensional sine algebra (39) [25]. For the rescaled generators  $\bar{T}_m = -\frac{\mathbf{i}}{2\pi\alpha} T_m$ , the  $SDiff(T^2)$  algebra (40) is recovered in the limit  $\alpha \rightarrow 0$ .

Since the product of the generators satisfies (69), according to theorem 5, it is known that the  $n$  algebra with respect to the generators (68) is (55). Thus in terms of the magnetic translation operators (68), we give an explicit physical realization of the (co)sine  $n$ -algebra (55).

Let us now discuss first few  $n$ -algebras.

- $[T_{i_1}, T_{i_2}, T_{i_3}] = 2\mathbf{i} [\sin(\pi\alpha(i_1 \wedge i_2 + i_1 \wedge i_3 + i_2 \wedge i_3)) - \sin(\pi\alpha(-i_1 \wedge i_2 + i_1 \wedge i_3 + i_2 \wedge i_3)) - \sin(\pi\alpha(i_1 \wedge i_2 + i_1 \wedge i_3 - i_2 \wedge i_3))] T_{i_1+i_2+i_3}$  (70)

Comparing (70) with (43), we see that when  $\alpha = \frac{1}{2}$ , (70) becomes a Fillipov 3-algebra. Let us take the rescaled generators  $\bar{T}_{i_j} = \sqrt{\frac{-\mathbf{i}}{2\pi\alpha}} T_{i_j}$ ,  $j = 1, 2, 3$ , we see that in the limit  $\alpha \rightarrow 0$ , (70) gives the  $SDiff(T^2)$  3-algebra (47).

- $[T_{i_1}, T_{i_2}, T_{i_3}, T_{i_4}]$   
 $= 2(\cos(\pi\alpha(i_1 \wedge i_2 + i_1 \wedge i_3 + i_1 \wedge i_4 + i_2 \wedge i_3 + i_2 \wedge i_4 + i_3 \wedge i_4))$   
 $- \cos(\pi\alpha(i_1 \wedge i_2 + i_1 \wedge i_3 + i_1 \wedge i_4 + i_2 \wedge i_3 + i_2 \wedge i_4 - i_3 \wedge i_4))$   
 $- \cos(\pi\alpha(i_1 \wedge i_2 + i_1 \wedge i_3 + i_1 \wedge i_4 - i_2 \wedge i_3 + i_2 \wedge i_4 + i_3 \wedge i_4))$   
 $+ \cos(\pi\alpha(i_1 \wedge i_2 + i_1 \wedge i_3 + i_1 \wedge i_4 - i_2 \wedge i_3 - i_2 \wedge i_4 + i_3 \wedge i_4))$   
 $+ \cos(\pi\alpha(i_1 \wedge i_2 + i_1 \wedge i_3 + i_1 \wedge i_4 + i_2 \wedge i_3 - i_2 \wedge i_4 - i_3 \wedge i_4))$   
 $- \cos(\pi\alpha(i_1 \wedge i_2 + i_1 \wedge i_3 + i_1 \wedge i_4 - i_2 \wedge i_3 - i_2 \wedge i_4 - i_3 \wedge i_4))$   
 $- \cos(\pi\alpha(-i_1 \wedge i_2 + i_1 \wedge i_3 + i_1 \wedge i_4 + i_2 \wedge i_3 + i_2 \wedge i_4 + i_3 \wedge i_4))$   
 $+ \cos(\pi\alpha(-i_1 \wedge i_2 + i_1 \wedge i_3 + i_1 \wedge i_4 + i_2 \wedge i_3 + i_2 \wedge i_4 - i_3 \wedge i_4))$   
 $+ \cos(\pi\alpha(-i_1 \wedge i_2 - i_1 \wedge i_3 + i_1 \wedge i_4 + i_2 \wedge i_3 + i_2 \wedge i_4 + i_3 \wedge i_4))$   
 $- \cos(\pi\alpha(-i_1 \wedge i_2 - i_1 \wedge i_3 - i_1 \wedge i_4 + i_2 \wedge i_3 + i_2 \wedge i_4 + i_3 \wedge i_4))$   
 $- \cos(\pi\alpha(-i_1 \wedge i_2 + i_1 \wedge i_3 - i_1 \wedge i_4 + i_2 \wedge i_3 + i_2 \wedge i_4 - i_3 \wedge i_4))$   
 $+ \cos(\pi\alpha(-i_1 \wedge i_2 - i_1 \wedge i_3 - i_1 \wedge i_4 + i_2 \wedge i_3 + i_2 \wedge i_4 - i_3 \wedge i_4))) T_{i_1+i_2+i_3+i_4}. \quad (71)$

For the cosine 4-algebra (71) with the arbitrary value  $\alpha \in R$ , it is a sh-4-Lie algebra. Taking the rescaled generators  $\bar{T}_{i_j} = \sqrt[3]{\frac{-i}{2\pi\alpha}} T_{i_j}$ ,  $j = 1, 2, 3, 4$ , in (71), then when  $\alpha \rightarrow 0$ , (71) becomes the null 4-algebra

$$[\bar{T}_{i_1}, \bar{T}_{i_2}, \bar{T}_{i_3}, \bar{T}_{i_4}] = 0. \quad (72)$$

• When  $n = 5$ , the corresponding 5-algebra is given by (56). Taking  $\alpha = \frac{1}{3}$  in (56), it immediately gives a sh-5-Lie algebra. Let us take rescaled generators  $\bar{T}_{i_j} = \sqrt[4]{\frac{-i}{2\pi\alpha}} T_{i_j}$ ,  $j = 1, 2, \dots, 5$ , in (56), then we have the null 5-algebra in the limit  $\alpha \rightarrow 0$ ,

$$[\bar{T}_{i_1}, \bar{T}_{i_2}, \bar{T}_{i_3}, \bar{T}_{i_4}, \bar{T}_{i_5}] = 0. \quad (73)$$

Not as the case of the sine 3-algebra (70), we note that when  $\alpha \rightarrow 0$ , the cosine 4-algebra (71) and 5-algebra (56) with respect to the rescaled generators become the null 4 and 5-algebras, respectively. For the (co)sine  $n$ -algebra (55) with  $n \geq 4$ , taking the rescaled generators  $\bar{T}_{i_j} = \sqrt[n-1]{\frac{-i}{2\pi\alpha}} T_{i_j}$ ,  $j = 1, 2, \dots, n$ , it is easy to verify that in the limit  $\alpha \rightarrow 0$ , (55) gives the null  $n$ -algebra (60).

## 6 Concluding Remarks

We have investigated the  $q$ -deformation of the infinite-dimensional  $n$ -algebras. The V-W algebra is the centerless Virasoro algebra. Its  $q$ -deformation has been well investigated in the literature. One has already known that in the usual way, the V-W  $n$ -algebra is null. In this paper, we firstly investigated the  $q$ -deformation of the null V-W  $n$ -algebra and constructed the nontrivial  $q$ -deformed V-W  $n$ -algebra. We found that it satisfies the sh-Jacobi's identity, but the FI fails. Thus this  $q$ -deformed V-W  $n$ -algebra is indeed a sh- $n$ -Lie algebra. Furthermore in terms of the pseud-differential operators on the quantum plane, we constructed the  $q$ -deformed  $SDiff(T^2)$   $n$ -algebra and proved that the sh-Jacobi's identity holds for even  $n$ . We also presented the (co)sine  $n$ -algebra which is the sh- $n$ -Lie algebra for the case of even  $n$ . The interesting cases are for  $n = 3$  and 5. We found that there exists a sine 3-algebra which is indeed a Fillipov 3-algebra. When  $n = 5$ , we derived a cosine 5-algebras which is a sh-5-Lie algebras. An interesting open question is whether there exists the special values such that the (co)sine  $n$ -algebra with general odd  $n$  is the Fillipov  $n$ -algebra or sh- $n$ -Lie algebra.

It is worthwhile to mention that we introduce the appropriate star product into the  $q$ - $n$ -bracket. This star product plays an important role in deriving the desired  $q$ -deformed V-W  $n$ -algebra. For the case of the  $q$ -deformed  $SDiff(T^2)$   $n$ -algebra, we tried to introduce the appropriate star product such that the sh-Jacobi's identity holds for odd  $n$ . Unfortunately, we did not succeed in finding any one. Whether there exists such kind of the star product still deserves further study.

Our investigation revealed a deep connection between the  $q$ -deformed infinite-dimensional  $n$ -algebra and the sh- $n$ -Lie algebra. One may construct the sh- $n$ -Lie algebra from the  $q$ -deformation point of view. It sheds new light on the sh- $n$ -Lie algebra. It would be interesting to study further and see whether there exist the central extension terms for the sh- $n$ -Lie algebra derived in this paper. Finally, it is worth to emphasize that we give an explicit physical realization of the (co)sine  $n$ -algebra in terms of the so-called magnetic translation operators. We believe that the application of the  $q$ -deformed V-W and  $SDiff(T^2)$   $n$ -algebras in physics should also be of interest.

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## References

- [1] T. Curtright and C. Zachos, Deforming maps for quantum algebras, Phys. Lett. B 243 (1990) 237.
- [2] N. Aizawa and H.Sato,  $q$ -deformation of the Virasoro algebra with central extension, Phys. Lett. B 256 (1991) 185.
- [3] M. Chaichian and P. Prešnajder, Sugawara construction and the  $q$ -deformation of Virasoro (super) algebra, Phys. Lett. B 277 (1992) 109.
- [4] J. Shiraishi, H. Kubo, H. Awata and S. Odake, A Quantum Deformation of the Virasoro Algebra and the Macdonald Symmetric Functions, Lett. Math. Phys. 38 (1996) 33 [arXiv:q-alg/9507034].
- [5] M. Chaichian, D. Ellinas and Z. Popowicz, Quantum conformal algebra with central extension, Phys. Lett. B 248 (1990) 95.
- [6] M. Chaichian, P. Kulish and J. Lukierski,  $q$ -deformed Jacobi identity,  $q$ -oscillators and  $q$ -deformed infinite-dimensional algebras, Phys Lett B 237 (1990) 401.
- [7] A.P. Polychronakos, Consistency conditions and representations of a  $q$ -deformed Virasoro algebra, Phys Lett B 256 (1991) 35.
- [8] M. Chaichian, A.P. Isaev, J. Lukierski, Z. Popowicz and P. Prešnajder,  $q$ -deformations of Virasoro algebra and conformal dimensions, Phys. Lett. B 262 (1991) 32.

- [9] Ch. Devchand and M.V. Saveliev, Comultiplication for quantum deformations of the centreless Virasoro algebra in the continuum formulation, *Phys. Lett. B* 258 (1991) 364.
- [10] H. Sato, Realizations of  $q$ -Deformed Virasoro Algebra, *Prog. Theor. Phys.* 89 (1993) 531.
- [11] J.L. Gervais and A. Neveu, Dual string spectrum in Polyakov's quantization (II). Mode separation, *Nucl. Phys. B* 209 (1982) 125.
- [12] J.L. Gervais, Infinite family of polynomial functions of the Virasoro generators with vanishing Poisson brackets, *Phys. Lett. B* 160 (1985) 277.
- [13] M. Chaichian, Z. Popowicz and P. Prešnajder,  $q$ -Virasoro algebra and its relation to the  $q$ -deformed KdV system, *Phys. Lett. B* 249 (1990) 63.
- [14] V.V. Bazhanov, S.L. Lukyanov and A.B. Zamolodchikov, Integrable Structure of Conformal Field Theory, Quantum KdV Theory and Thermodynamic Bethe Ansatz, *Commun. Math. Phys.* 177 (1996) 381 [arXiv:hep-th/9412229].
- [15] D. Fioravanti and M. Rossi, A braided Yang-Baxter algebra in a Theory of two coupled Lattice Quantum KdV: algebraic properties and ABA representations, *J. Phys. A: Math. Theor.* 35 (2002) 3647 [arXiv:hep-th/0104002].
- [16] S. Lukyanov, A note on the deformed Virasoro algebra, *Phys. Lett. B* 367 (1996) 121 [arXiv:hep-th/9509037].
- [17] D. Fioravanti and M. Rossi, The elliptic scattering theory of the 1/2-XYZ and higher order deformed Virasoro algebras, *Ann. Henri Poincaré* 7 (2006) 1449 [arXiv:hep-th/0602080].
- [18] C.Z. Zha,  $q$ -deformation of highorder Virasoro algebra, *J. Math. Phys.* 35 (1994) 517.
- [19] C.N. Pope, L.J. Romans and X. Shen, The complete structure of  $W_\infty$ , *Phys. Lett. B* 236 (1990) 173.
- [20] E. Floratos and J. Iliopoulos, A note on the classical symmetries of the closed bosonic membranes, *Phys. Lett. B* 201 (1988) 237.
- [21] I. Antoniadis, P. Ditsas, E. Floratos and J. Iliopoulos, New realizations of the Virasoro algebra as membrane symmetries, *Nucl. Phys. B* 300 (1988) 549.
- [22] E.H. El Kinani and M. Zakkari, On the  $q$ -deformation of certain infinite dimensional Lie algebras, *Phys. Lett. B* 357 (1995) 105.
- [23] D.B. Fairlie and C.K. Zachos, Infinite-dimensional algebras, sine brackets, and  $SU(\infty)$ , *Phys. Lett. B* 224 (1989) 101.

- [24] D.B. Fairlie, P. Fletcher and C.K. Zachos, Trigonometric structure constants for new infinite-dimensional algebras, Phys. Lett. B 218 (1989) 203.
- [25] T. Dereli and A. Vermin, A physical realisation of the super-sine algebra, Phys. Lett. B 288 (1992) 109.
- [26] A. Jellal, M. Daoud and Y. Hassouni, Supersymmetric sine algebra and degeneracy of Landau levels, Phys. Lett. B 474 (2000) 122.
- [27] Y. Nambu, Generalized Hamiltonian dynamics, Phys. Rev. D 7 (1973) 2405.
- [28] L. Takhtajan, On Foundation of the generalized Nambu mechanics, Commun. Math. Phys. 160 (1994) 295 [arXiv:hep-th/9301111].
- [29] J. Bagger and N. Lambert, Modeling multiple M2's, Phys. Rev. D 75 (2007) 045020 [arXiv:hep-th/0611108].
- [30] J. Bagger and N. Lambert, Gauge symmetry and supersymmetry of multiple M2-branes, Phys. Rev. D 77 (2008) 065008 [arXiv:0711.0955].
- [31] A. Gustavsson, Algebraic structures on parallel M2-branes, Nucl. Phys. B 811 (2009) 66 [arXiv:0709.1260].
- [32] T.L. Curtright, D.B. Fairlie and C.K. Zachos, Ternary Virasoro-Witt algebra, Phys. Lett. B 666 (2008) 386 [arXiv:0806.3515].
- [33] T. Curtright, D. Fairlie, X. Jin, L. Mezincescu and C. Zachos, Classical and quantal ternary algebras, Phys. Lett. B 675 (2009) 387 [arXiv:0903.4889].
- [34] H. Lin, Kac-Moody extensions of 3-algebras and M2-branes, JHEP 07 (2008) 136 [arXiv:0805.4003].
- [35] S. Chakraborty, A. Kumar and S. Jain,  $w_\infty$  3-algebra, JHEP 09 (2008) 091 [arXiv:0807.0284].
- [36] M.R. Chen, K. Wu and W.Z. Zhao, Super  $w_\infty$  3-algebra, JHEP 09 (2011) 090 [arXiv:1107.3295].
- [37] M.R. Chen, S.K. Wang, K. Wu and W.Z. Zhao, Infinite-dimensional 3-algebra and integrable system, JHEP 12 (2012) 030 [arXiv:1201.0417[nlin.SI]].
- [38] M.R. Chen, S.K. Wang, X.L. Wang, K. Wu and W.Z. Zhao, On  $W_{1+\infty}$  3-algebra and integrable system, Nucl. Phys. B 891 (2015) 655 [arXiv:1309.4627[nlin.SI]].

- [39] F. Ammar, A. Makhoul and S. Silvestrov, Ternary  $q$ -Virasoro-Witt Hom-Nambu-Lie algebras, J. Phys. A: Math. Theor. 43 (2010) 265204.
- [40] V.T. Filippov,  $n$ -Lie algebras, Sib. Math. J. 26 (1985) 879.
- [41] M. Goze, N. Goze and E. Remm,  $n$ -Lie algebras, African J. Math. Phys. 8 (2010) 17 [arXiv:0909.1419[math.RA]].
- [42] J.A. de Azcárraga and J.M. Izquierdo,  $n$ -ary algebras: a review with applications, J. Phys. A: Math. Theor. 43 (2010) 293001.
- [43] A.J. MacFarlane, On  $q$ -analogues of the quantum harmonic oscillator and the quantum group  $SU(2)_q$ , J. Phys. A: Math. Gen. 22 (1989) 4581.
- [44] L.C. Biedenharn, The quantum group  $SU_q(2)$  and a  $q$ -analogue of the boson operators, J. Phys. A: Math. Gen. 22 (1989) L873.
- [45] T. Hayashi,  $Q$ -analogues of Clifford and Weyl algebras-Spinor and oscillator representations of quantum enveloping algebras, Commun. Math. Phys. 127 (1990) 129.
- [46] T. Brzeziński, H. Da.browski and J. Rembieliński, On the quantum differential calculus and the quantum holomorphicity, J. Math. Phys. 33 (1992) 19.